On collisions with unlimited energies in the vicinity of Kerr and Schwarzschild black hole horizons

A. A. Grib[1], **Yu. V. Pavlov**[2] **and O. F. Piattella**[3]

[1,2] *A.Friedmann Laboratory for Theoretical Physics, Griboedov kanal 30/32, St. Petersburg 191023, Russia*
[1] *Theoretical Physics and Astronomy Department, The Herzen University, 48, Moika, St. Petersburg, 191186, Russia*
[2] *Institute of Mechanical Engineering, Russian Acad. Sci., Bol'shoy pr. 61, St. Petersburg 199178, Russia*
[3] *Dep. de Física, Universidade Federal do Espírito Santo, av. Ferrari 514, 29075-910 Vitória, ES, Brazil*
[3] *INFN sezione di Milano, Via Celoria 16, 20133 Milano, Italy*

Two particle collisions close to the horizon of the rotating nonextremal Kerr's and Schwarzschild black holes are analyzed. For the case of multiple collisions it is shown that high energy in the centre of mass frame occurs due to a great relative velocity of two particles and a large Lorentz factor. The dependence of the relative velocity on the distance to horizon is analyzed, the time of movement from the point in the accretion disc to the point of scattering with large energy as well as the time of back movement to the Earth are calculated. It is shown that they have reasonable order.

1 Introduction

There is much interest today to the high energy processes in the ergosphere of the Kerr's rotating black hole as the model for Active Galactic Nuclei (AGN). In Ref. [1] some of the authors of this paper put the hypothesis that due to Penrose process and scattering in the vicinity of the horizon superheavy particles of dark matter due to the large centre of mass energy transfer can become ordinary particles observed on the Earth as ultra high energy cosmic rays (UHECR) by the AUGER group [2].

In Ref. [3] a resonance for the centre of mass (CM) energy of two scattering particles close to the horizon of the extremal Kerr's black hole was found. Let us call this effect the BSW effect. In our papers [4]– [7] it was shown that the BSW effect can occur for the nonextremal black hole if one takes into account the possibility of multiple scattering of the particle: in the first scattering close to the horizon the particle gets the angular momentum close to the critical one. In the second scattering close to the first one the particles due to BSW effect occur to be in the region of high energy physics — Grand Unification or even Planckean physics. In Ref. [8] (see also Refs. [9,10]) it was shown that the BSW effect can be connected with the special behaviour of the Killing vector on the ergosphere and the large Lorentz factor for relative velocity of two particles. In this paper we continue our analysis of this process made in Refs. [4]– [7].

The system of units $G=c=1$ is used in the paper.

[1] e-mail: andrei_grib@mail.ru
[2] e-mail: yuri.pavlov@mail.ru
[3] e-mail: oliver.piattella@gmail.com

2 The scattering energy in the centre of mass frame

The Kerr's metric of the rotating black hole in Boyer–Lindquist coordinates has the form

$$ds^2 = dt^2 - \frac{2Mr}{\rho^2}(dt - a\sin^2\theta\, d\varphi)^2 - \rho^2\Big(\frac{dr^2}{\Delta} + d\theta^2\Big) - (r^2+a^2)\sin^2\theta\, d\varphi^2, \qquad (1)$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 - 2Mr + a^2, \qquad (2)$$

M is the mass of the black hole, aM is angular momentum. The event horizon for the Kerr's black hole corresponds to the value

$$r = r_H \equiv M + \sqrt{M^2 - a^2}. \qquad (3)$$

The Cauchy horizon is

$$r = r_C \equiv M - \sqrt{M^2 - a^2}. \qquad (4)$$

For geodesics in Kerr's metric (1) one obtains (see Ref. [11], Sec. 3.4.1)

$$\rho^2 \frac{dt}{d\lambda} = -a\left(aE\sin^2\theta - J\right) + \frac{r^2+a^2}{\Delta}\, P, \qquad (5)$$

$$\rho^2 \frac{d\varphi}{d\lambda} = -\left(aE - \frac{J}{\sin^2\theta}\right) + \frac{aP}{\Delta}, \tag{6}$$

$$\rho^4 \left(\frac{dr}{d\lambda}\right)^2 = R, \quad \rho^4 \left(\frac{d\theta}{d\lambda}\right)^2 = \Theta. \tag{7}$$

Here m is the rest mass of the probe particle, $\lambda = \tau/m$, where τ is the proper time for massive particle, E is conserved energy of the probe particle, J is conserved angular momentum projection on the rotation axis of the black hole,

$$P = \left(r^2 + a^2\right) E - aJ, \tag{8}$$

$$R = P^2 - \Delta[m^2 r^2 + (J - aE)^2 + Q], \tag{9}$$

$$\Theta = Q - \cos^2\theta \left[a^2(m^2 - E^2) + \frac{J^2}{\sin^2\theta}\right], \tag{10}$$

Q is the Carter's constant. For massless particle one must take the limit $m \to 0$ in formulas (9), (10). For equatorial ($\theta = \pi/2$) geodesics the Carter constant is equal zero and $\Theta = 0$ also.

Let us find the energy $E_{\rm c.m.}$ in the centre of mass system of two colliding particles with rest masses m_1 and m_2 in arbitrary gravitational field. It can be obtained from

$$(E_{\rm c.m.}, 0\,, 0\,, 0\,) = p^i_{(1)} + p^i_{(2)}, \tag{11}$$

where $p^i_{(n)}$ is 4-momentum of particle with number n. Taking the squared (11) and due to $p^i_{(n)} p_{(n)i} = m_n^2$ one obtains

$$E^2_{\rm c.m.} = m_1^2 + m_2^2 + 2 p^i_{(1)} p_{(2)i}. \tag{12}$$

The scalar product does not depend on the choice of the coordinate frame so (12) is valid in an arbitrary coordinate system and for arbitrary gravitational field.

Note that the transformation to the centre of mass coordinate system is always possible excluding the case of two massless particles with identically directional momenta. But in this case particles cannot collide.

Let us find the expression for the collision energy of two particles freely falling at the equatorial plane of the rotating black hole. We denote $x = r/M\,,\;\; A = a/M\,,\;\; j_n = J_n/M\,,$

$$x_H = 1 + \sqrt{1 - A^2}, \quad x_C = 1 - \sqrt{1 - A^2}, \tag{13}$$

$$\Delta_x = x^2 - 2x + A^2 = (x - x_H)(x - x_C). \tag{14}$$

Using (5)–(7) one obtains:

$$\begin{aligned} p^i_{(1)} p_{(2)i} = \frac{1}{x \Delta_x} \Bigg\{ & E_1 E_2 \left(x^3 + A^2(x+2)\right) - \\ & - 2A\left(j_1 E_2 + j_2 E_1\right) + j_1 j_2 (2 - x) - \\ & - \Bigg[\Big(2E_1^2 x^2 + 2(j_1 - E_1 A)^2 - j_1^2 x \\ & + (E_1^2 - m_1^2) x \Delta_x \Big) \Big(2E_2^2 x^2 + 2(j_2 - E_2 A)^2 \\ & - j_2^2 x + (E_2^2 - m_2^2) x \Delta_x \Big) \Bigg]^{\frac{1}{2}} \Bigg\}. \end{aligned} \tag{15}$$

The value of Δ_x is going to zero on the event horizon and as it is seen from (15) the scalar product of four vectors $u^i_{(1)} u_{(2)i}$ and the collision energy of particles on the horizon can be divergent depending on the behavior of the denominator of the formula. To find the limit $r \to r_H$ for the black hole with a given angular momentum A one must take in (15) $x = x_H + \alpha$ with $\alpha \to 0$ and do calculations up to the order α^2. Taking into account $A^2 = x_H x_C$, $x_H + x_C = 2$, after resolution of uncertainties in the limit $\alpha \to 0$ one obtains

$$\begin{aligned} E^2_{\rm c.m.}(r \to r_H) = & \frac{(J_{1H} J_2 - J_{2H} J_1)^2}{4M^2 (J_{1H} - J_1)(J_{2H} - J_2)} \\ & + m_1^2 \left[1 + \frac{J_{2H} - J_2}{J_{1H} - J_1}\right] + m_2^2 \left[1 + \frac{J_{1H} - J_1}{J_{2H} - J_2}\right], \end{aligned} \tag{16}$$

where

$$J_{nH} = \frac{2 E_n r_H}{A} = \frac{E_n}{\Omega_H}\,, \tag{17}$$

$\Omega_H = A/2r_H$ is horizon angular velocity [11]. Formula (16) can be used when one or both particles become massless. The BSW effect can be considered also in this case. In [7, 12] the expression of $E_{\rm c.m.}$ is written in other forms.

For the Schwarzschild black hole ($A = 0$) the energy of collision in the centre of mass frame is

$$\begin{aligned} E^2_{\rm c.m.}(r \to r_H) = & \frac{(E_1 J_2 - E_2 J_1)^2}{4M^2 E_1 E_2} \\ & + m_1^2 \left(1 + \frac{E_2}{E_1}\right) + m_2^2 \left(1 + \frac{E_1}{E_2}\right). \end{aligned} \tag{18}$$

As it can be seen from (16) the collision energy of particle in the centre of mass frame goes to infinity on the horizon if the angular momentum of one of the freely falling particles has the value J_{nH}. Is falling of the particle with such a value of the angular momentum on the horizon possible? For the

case of the free fall from infinity on the nonextremal $A < 1$ rotating black hole it is impossible. This can be seen from the fact that the expression (9) on the horizon is going to zero for $J \to J_H$ but it's derivative with respect to r for $A < 1$ is negative.

Consider the case of the collision of two massive particles. For massive particles $p^i_{(n)} = m_{(n)} u^i_{(n)}$, where $u^i = dx^i/d\tau$. In this case, as one can see from (12), the energy $E_{\rm c.m.}$ has maximal value for given $u_{(1)}, u_{(2)}$ and $m_1 + m_2$, if the particle masses are equal: $m_1 = m_2$.

Let us find the expression of the energy in the centre of mass frame through the relative velocity $v_{\rm rel}$ of particles at the moment of collision [13]. In the reference frame of the first particle one has for the components of 4-velocities of particles at this moment

$$u^i_{(1)} = (1, 0, 0, 0), \quad u^i_{(2)} = \frac{(1, \mathbf{v}_{\rm rel})}{\sqrt{1 - v^2_{\rm rel}}}. \qquad (19)$$

So $u^i_{(1)} u_{(2)i} = 1 \Big/ \sqrt{1 - v^2_{\rm rel}}$,

$$v_{\rm rel} = \sqrt{1 - \left(u^i_{(1)} u_{(2)i} \right)^{-2}}. \qquad (20)$$

These expressions evidently don't depend on the coordinate system.

From (12) and (20) one obtains

$$E^2_{\rm c.m.} = m_1^2 + m_2^2 + \frac{2 m_1 m_2}{\sqrt{1 - v^2_{\rm rel}}} \qquad (21)$$

and the nonlimited growth of the collision energy in the centre of mass frame occurs due to growth of the relative velocity to the velocity of light [10].

The massive particle free falling in the black hole with dimensionless angular momentum A being nonrelativistic at infinity ($E = m$) to achieve the horizon of the black hole must have angular momentum from the interval (l_L, l_R),

$$l_L = -2[1 + \sqrt{1 + A}], \quad l_R = 2[1 + \sqrt{1 - A}] \qquad (22)$$

(here the notation $l = J/mM$ is used). For $A < 1$ one has $l_R < l_H$ and even for values close to the extremal $A = 1$ of the rotating black hole $E^{\max}_{\rm c.m.}/\sqrt{m_1 m_2}$ can be not very large as mentioned in Refs. [14], [15] for the case $m_1 = m_2$. So for $A_{\max} = 0.998$ considered as the maximal possible dimensionless angular momentum of the astrophysical black holes (see Ref. [16]) one obtains $E^{\max}_{\rm c.m.}/\sqrt{m_1 m_2} \approx 18.97$. However this evaluation is enough for collisions of superheavy particles of dark matter with the mass close to the Grand Unification scale to occur in the region of Grand Unification interaction physics so that these particles can decay on quarks and be observed as the UHECR [1].

Does it mean that in processes of usual particles (protons, electrons) scattering in the vicinity of the rotating nonextremal black holes the scattering energy is limited so that no Grand Unification or even Planckean energies can be obtained? As it was shown for the first time in [4] if one takes into account the possibility of multiple scattering so that the particle falling from infinity on the black hole with some fixed angular momentum changes its momentum in the result of interaction with particles in the accreting disc and after this is again scattering close to the horizon then the scattering energy can be unlimited.

Note that the critical values of J_H have various values for different values of the energies E (see (17)). That is why there is not only one critical trajectory for collision with unlimited energy but a set of trajectories corresponding to an interval of specific energies of particles falling onto a black hole.

Let us get the expression for the permitted interval in r for particles with angular momentum $l = l_H - \delta$ close to the horizon. From (7), (8), (9) one has for the movement of massive particle in the equatorial plane

$$\left(\frac{dr}{d\tau} \right)^2 = \varepsilon^2 + \frac{2}{x^3} (A\varepsilon - l)^2 + \frac{A^2 \varepsilon^2 - l^2}{x^2} - \frac{\Delta_x}{x^2}, \qquad (23)$$

where $\varepsilon = E/m$ is specific energy of particle. As it was shown in Ref. [17], Sec. 88, the specific energy in the static gravitational field is equal to

$$\varepsilon = \sqrt{\frac{g_{00}}{1 - \mathbf{v}^2}}, \qquad (24)$$

where $\mathbf{v}$ is the velocity of the particle measured by the observer at rest at the point of the passing particle.

To get the boundaries of the permitted interval in x one must put the left hand side of (23) to zero and find the root. In the second order in δ close to the horizon one obtains

$$x_\delta \approx x_H + \frac{\delta^2 x_C^2}{x_H (x_H - x_C)(\varepsilon^2 x_H + x_C)}. \qquad (25)$$

Note that smallness of the value of $x_\delta - x_H$ does not mean the smallness of the "physical distance" from r_δ to horizon (see Ref. [17], Sec. 84).

From (15), (25) one obtains for values of function $u^i_{(1)} u_{(2)i}$ for $l_1 = l_{1H} - \delta$ after first scattering in points x_δ and on the horizon

$$u^i_{(1)} u_{(2)i}(x_\delta) \approx \frac{(l_{2H} - l_2)(\varepsilon_1^2 x_H + x_C)}{\delta \cdot x_C}, \qquad (26)$$

$$u^i_{(1)} u_{(2)i}(x_H) \approx \frac{(l_{2H} - l_2)(\varepsilon_1^2 x_H + x_C)}{2\delta \cdot x_C}. \qquad (27)$$

Therefore the function $u^i_{(1)} u_{(2)i}$ and hereby the energy of collisions decrease near the horizon! The dependence of $u^i_{(1)} u_{(2)i}$ on the coordinate r is shown on Fig. 1.

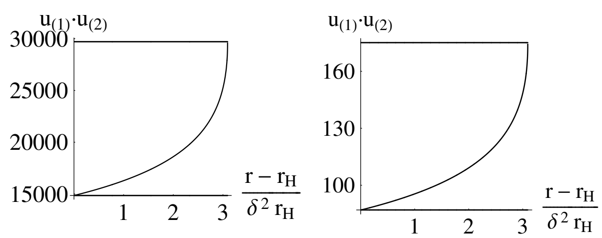

Figure 1: The dependence of $u^i_{(1)} u_{(2)i}$ on the coordinate r for $A = 0.998$, $\delta = 5 \cdot 10^{-4}$, $\varepsilon_1 = \varepsilon_2 = 1$, $l_2 = l_L$ on the left and $l_2 = l_R$ on the right

The left picture shows that the collision energy can be very large immediately after obtaining the angular momentum $l_H - \delta$. But, for $A_{\max} = 0.998$, $l_H - l_R \approx 0.04$ and so the collision energy for particles with angular momentum l_R is not large. This means that from the decrease of the collision energy with fixed value of angular momentum does not follow the extremely large energy of particles in the centre of mass frame needed to get the value $l_H - \delta$ in the intermediate collision.

The decrease of the collision energy in the centre of mass frame of the free falling particle and the particle with critical angular momentum $l = l_H - \delta$ in their movement to horizon is explained by the decrease of the relative velocity of these particles when going from the point x_δ. Due to the definition (25) the radial velocity of the particle with the angular momentum $l = l_H - \delta$ is equal to zero: $dr/d\tau = 0$ and $dr/dt = 0$. But the other noncritical particle colliding with the critical one has large value of the radial velocity. Angular velocities $d\varphi/dt$ for both particles go to the same value Ω_H. So large relative velocity of two particles occurs due to a "stop" of the critical particle in radial direction. But after this both particles increase their radial velocities $dr/d\tau$ and the relative velocity is decreasing — the critical particle is "running down" the free falling one. From (20), (26), (27) one obtains for the relative velocity of the particles colliding at the point x_δ and on the horizon

$$1 - v_{\rm rel}(x_\delta) = \frac{\delta^2 x_C^2}{2\left(l_{2H} - l_2\right)^2 \left(\varepsilon_1^2 x_H + x_C\right)^2}, \qquad (28)$$

$$1 - v_{\rm rel}(x_H) = \frac{2\delta^2 x_C^2}{\left(l_{2H} - l_2\right)^2 \left(\varepsilon_1^2 x_H + x_C\right)^2}. \qquad (29)$$

So the physical reason of the unlimited great energy of the collision in the centre of mass frame of the particles falling in the black hole is the increasing of the relative velocity of particles at the moment of collision to the velocity of light. So one can expect very large energy of collision in the case when one of the particles due to multiple intermediate collisions in the accretion disc strongly diminishes its energy so that its velocity becomes small near the horizon. Really from (16) it is easy to obtain that

$$\frac{E_{\rm c.m.}}{\sqrt{m_1 m_2}} \sim \sqrt{\frac{l_{2H} - l_2}{l_{1H} - l_1}} \to \infty\,, \quad \varepsilon_1, l_1 \to 0\,. \qquad (30)$$

From the same considerations one can conclude that for the case of nonrotating Schwarzschild black hole the collision energy of the free falling particle with the particle at rest close to horizon also is great and unlimited. Using (12), (15), (24) for $A = 0$ and the particle at rest in the point with radial coordinate r_0 (so $l_1 = 0$, $\varepsilon_1 = \sqrt{1 - r_g/r_0}$, $dr_1/d\tau = 0$) for the energy of its collision with the particle with ε_2, l_2 one obtains in the centre of mass frame

$$E^2_{\rm c.m.} = m_1^2 + m_2^2 + 2 m_1 E_2 \sqrt{\frac{r_0}{r_0 - r_g}}, \qquad (31)$$

which evidently is growing infinitely for $r_0 \to r_g$.

Note that the stopping of one particle on the horizon of a nonrotating charged black hole takes place for its critical charge and then it follows to expect the infinity energy of collisions as it was shown in [18].

If one particle in the point with radial coordinate r_0 has $dr/d\tau = 0$ but $l_1 \neq 0$, then from (12),

(15), (23) one has

$$E^2_{\rm c.m.} = m_1^2 + m_2^2 + 2m_1m_2\left[\varepsilon_2\sqrt{\frac{l_1^2+x_0^2}{(x_0-2)x_0}} - \frac{l_1 l_2}{x_0^2}\right], \quad (32)$$

which also is growing infinitely for $x_0 \to x_H = 2$.

Note that for particles nonrelativistic on infinity with $m_1 = m_2 = m$, freely falling on the Schwarzschild black hole the limiting energy of collisions is only $2\sqrt{5}m$ (see Ref. [19]).

In conclusion of this part note that the probability of the collision of the relativistic particle with the particle at rest close to the Schwarzschild horizon is very small. So this is the main difference with the situation when the BSW resonance occurs. This can be seen from the evaluation of the interval of the proper time of falling from the point r_0 where $dr/d\tau = 0$ to horizon. Define the effective potential through the right hand side of (23)

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{\rm eff} = 0. \quad (33)$$

Then

$$\left.\frac{dr}{d\tau}\right|_{r_0} = 0 \quad \Rightarrow \quad r \approx r_0 - \frac{\Delta\tau^2}{2}\frac{dV_{\rm eff}}{dr}. \quad (34)$$

So the proper time of falling of the particle to horizon is

$$\Delta\tau \approx M\sqrt{2(x_0 - x_H)\bigg/\frac{dV_{\rm eff}}{dx}}. \quad (35)$$

For the Schwarzschild black hole one obtains

$$\Delta\tau \approx 4M\sqrt{\frac{2(x_0 - x_H)}{4 + l^2}}, \quad x_0 \to x_H. \quad (36)$$

For the Kerr black hole taking l close to l_H from (23), (33), (35) one obtains

$$\Delta\tau \approx MA\sqrt{\frac{2x_H(x_0 - x_H)}{\sqrt{1-A^2}\,(\varepsilon^2 x_H + x_C)}}, \quad x_0 \to x_H, \quad (37)$$

which evidently is much larger than (36) for $A \to 1$.

3 Estimate of the time of the fall before collision leading to the large energy

In Refs. [4]– [6] it was shown by us that in order to get the unboundedly growing energy for the extremal case one must have the time interval (as coordinate as proper time) from the beginning of the falling inside the black hole to the moment of collision also growing infinitely. Quantitative estimations have been given for a case of extremely rotating black hole $A = 1$ [5, 7]. Here we consider the case $A < 1$.

From Eqs. (5), (23) one gets

$$|dt/dx| = \frac{M\sqrt{x}\left((x^3 + A^2x + 2A^2)\varepsilon - 2Al\right)}{\Delta_x\sqrt{2\varepsilon^2x^2 - l^2x + 2(A\varepsilon - l)^2 + (\varepsilon^2 - 1)\Delta_x}}. \quad (38)$$

For $A < 1$ from (14), (38) one gets that the value of the time interval measured by clock of the distant observer necessary to achieve the horizon is logarithmically divergent. From (38) one has for $l < l_H = 2\varepsilon x_H/A$, $A < 1$ and x_0 close to x_H (for example $x_0 = 2x_H$)

$$\Delta t \sim -\frac{2Mx_H}{x_H - x_C}\log(x_f - x_H), \quad x_f \to x_H. \quad (39)$$

Remind that for the extremal black hole and the critical value of the angular momentum of the falling particle this interval is divergent as $1/(r - r_H)$.

From Eqs. (12), (25), (26), (27), (39) it is easy to obtain for collisions of two particles with $l_1 = l_H - \delta$ close to horizon at the point x_δ

$$\Delta t \sim \frac{8Mx_H}{x_H - x_C}\,\log\frac{E_{\rm c.m.}}{\sqrt{m_1m_2}}. \quad (40)$$

So for

$$A = 0.998, \quad \Delta t \sim 3.2\cdot 10^{-4}\frac{M}{M_\odot}\log\frac{E_{\rm c.m.}}{\sqrt{m_1m_2}}\,{\rm s.} \quad (41)$$

Taking the value of the Grand Unification energy $E_{\rm c.m.}/\sqrt{m_1m_2} = 10^{14}$ and the mass of the black hole $10^8 M_\odot$ typical for Active Nuclei of galaxies one obtains $\Delta t \sim 10^6$ s, i.e. of the order of 12 days. So in case of the nonextremal rotating black hole the mechanism of the intermediate collision to get the additional angular momentum with the following collision with other relativistic particle leading to large collision energy proposed by us in Ref. [4] needs reasonable time much smaller than that for the extremal case.

One can ask about the time of back movement of the particle after collision with very high energy from the vicinity of horizon to the Earth. Due to reversibility of equations of motion in time it is easy to see that this time is equal to the sum of the

same 12 days to accretion disc and some 10–100 megaparsec — the distance of the AGN from the Earth.

Acknowledgments. One of the authors A.A.G. is indebted to CAPES for financial support of the work on the first part of the paper and to the UFES, Brazil, for hospitality, other author O.F.P. is thankful to CNPq for partial financial support of his work.

References

[1] A. A. Grib and Yu. V. Pavlov, Mod. Phys. Lett. A **23**, 1151 (2008).

[2] The Pierre Auger Collaboration, Science **318**, 938 (2007); Astropart. Phys. **34**, 314 (2010).

[3] M. Banados, J. Silk and S. M. West, Phys. Rev. Lett. **103**, 111102 (2009).

[4] A. A. Grib and Yu. V. Pavlov, Pis'ma v ZhETF **92**, 147 (2010) [JETP Lett. **92**, 125 (2010)].

[5] A. A. Grib and Yu. V. Pavlov, Astropart. Phys. **34**, 581 (2011).

[6] A. A. Grib and Yu. V. Pavlov, *On Particle Collisions near Kerrs Black Holes*, arXiv: 1007.3222.

[7] A. A. Grib and Yu. V. Pavlov, Grav. Cosmol. **17**, 42 (2011).

[8] O. B. Zaslavskii, Class. Quantum Grav. **28**, 105010 (2011).

[9] O. B. Zaslavskii, Phys. Rev. D **82**, 083004 (2010); T. Harada and M. Kimura, Phys. Rev. D **83**, 024002 (2011).

[10] O. B. Zaslavskii, Phys. Rev. D **84**, 024007 (2011).

[11] V. P. Frolov and I. D. Novikov, *Black Hole Physics: Basic Concepts and New Developments* (Kluwer Acad. Publ., Dordrecht, 1998).

[12] T. Harada and M. Kimura, Phys. Rev. D **83**, 084041 (2011).

[13] M. Banados, B. Hassanain, J. Silk and S. M. West, Phys. Rev. D **83**, 023004 (2011).

[14] E. Berti, V. Cardoso, L. Gualtieri, F. Pretorius and U. Sperhake, Phys. Rev. Lett. **103**, 239001 (2009).

[15] T. Jacobson and T. P. Sotiriou, Phys. Rev. Lett. **104**, 021101 (2010).

[16] K. S. Thorne, Astrophys. J. **191**, 507 (1974).

[17] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon Press, Oxford, 1983).

[18] O. B. Zaslavskii, Pis'ma v ZhETF **92** 635 (2010) [JETP Lett. **92** 571 (2010)].

[19] A. N. Baushev, Int. J. Mod. Phys. D **18**, 1195 (2009).